Lecture 12: Infinite Series

1 Series in Physics

In statistical mechanics, the theormodynamics of a system is described by a partition function, which is weighted sum over states,

$$Z = \sum_i \exp(-\beta E_i). \tag{1}$$

For discrete systems, such as an Ising spin model, the partition takes the form of an infinite series

Another application of series is to the solution of linear differential equations. For example, the solution $f(x)$ of a differential equation may be expanded in a power series,

$$f(x) = \sum_i a_i x^i. \tag{2}$$

Since convergent series are term-by-term by term differentiable, this substitution turns the differential equation into a system of algebraic equations for the coefficients a_i. Since differentiation can change the convergence properties of series, it is important to be able to determine the radius of convergence for solutions obtained this way.

Series are particularly important to complex analysis. In fact, it would make sense to begin covering complex variables in this section. To keep things simple we will just talk about real series now, but anticipate that all this will carry over to the complex plane.

2 Infinite Series and Limits

The partial sum S_N of a series is the sum of the first N terms. The sum S of a series is defined as the limit

$$S = \lim_{N\to\infty} S_N. \tag{3}$$

Formally, we require that for any $\epsilon > 0$, there is an M_ϵ such that all partial sums are within ϵ of the limit S, that is $|S - S_N| < \epsilon \qquad (N > M_\epsilon)$. In other words, the partial sums S_N cluster closer and closer to the limit S for larger N. A non-convergent series either diverges to $\pm\infty$ or oscillates (or both).

2.1 Absolute vs. Conditional Convergence

A series $S = \sum a_i$ is said to be absolutely convergence if the series $S' = \sum |a_i|$ converges. If a series converges, but does not converge absolutely, it is said to be conditionally convergent.

Absolute convergence is a much stronger requirement on a series, and is usually more desireable. For example, the value of an absolutely convergent series is unchanged if the terms are rearanged. The value of a conditionally convergent series depends on the order of terms. In fact, a conditionally convergent series can take on any value from $-\infty$ to $+\infty$ by rearranging terms.

3 Common Series

3.1 Geometric series

The geometric series has a fixed ratio r between terms,

$$S_N = a + ar + ar^2 + \cdot + ar^{N-1} = \sum_{n=0}^{N-1} ar^n. \tag{4}$$

The sum may be evaluated by subtracting rS_N,

$$S_N = a + ar + ar^2 + \cdot + ar^{N-1} \tag{5}$$

$$rS_N = ar + ar^2 + ar^3 + \cdot + ar^N \tag{6}$$

$$(1-r)S_N = a - a^r N \tag{7}$$

$$S_N = \frac{a(1-r^N)}{1-r}. \tag{8}$$

The infinite series, S, converges for $|r| < 1$,

$$S = \frac{a}{1-r}, \qquad |r| < 1. \tag{9}$$

3.2 Harmonic series

The harmonic series is

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}. \tag{10}$$

This series can be shown to diverge by grouping terms

$$S = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots, \tag{11}$$

$$S > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots. \tag{12}$$

Thus, even though the individual terms of the series tend to zero, the sum diverges. Comparison to the integral $\int_1^\infty x^{-1} dx$ shows that this is logrithmic divergence.

4 Convergence tests

4.1 A preliminary test — do terms tend to zero?

A neccessary requirement for convergence is that the terms tend to zero, $\lim_{n\to\infty} u_n = 0$. This is hardly a sufficient test for convergence, but is a quick way to identify clearly non-convergent series.

Unless the series is clearly divergent, then more careful tests must be applied. The following tests apply to series with all positive terms. Application of these tests to any particular series requires practice, some test are easier to apply than others. Other tests exist, but these are among the most common.

4.2 Comparison test

If each term in a series $\sum a_i$ is less than the corresponding term in a convergent series $\sum u_i$, that is $a_i \leq u_i$, then the series $\sum a_i$ also converges. Conversly, if each term in a series $\sum a_i$ is greater than the corresponding term in a divergent series $\sum v_i$, that is $a_i \geq u_i$, then the series $\sum a_i$ also diverges.

It is common to compare to a convergent harmonic series to show convergence, and to compare to the harmonic series to show non-convergence.

4.3 Cauchy Root test

A powerful way to test a series against the geometric series is to take roots of the terms. If $(a_n)^{1/n} < 1$ for all large n, then the series converges. If $(a_n)^{1/n} \geq 1$ for large n, then the series converges.

4.4 D'Alembert Ratio test

Another quick way to compare to the geometic series is to check the ratio a_{n+1}/a_n. The test is

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &< 1, \qquad \text{convergence,} \\ &> 1, \qquad \text{divergence,} \\ &= 1, \qquad \text{indeterminant.} \end{aligned} \tag{13}$$

Although this test is often very easy to apply, the indeterminant result requires additional tests.

4.5 Integral test

The integral test is a powerful way to study the convergence and approximate value of a *monotonically decreasing* series. If the terms of the series a_n are identified with a monotonically decreasing function $f(x)$ so that $f(n) = a_n$, then the series is bounded above and below by the value of the integral,

$$\int_1^\infty f(x)dx < \sum_{n=1}^\infty a_n < a_1 + \int_1^\infty f(x)dx. \tag{14}$$

Thus, the convergence or divergence of the series is the same as the convergence or divergence of the integral. Additionally, this technique places limits on the remainder of the series after N terms have been summed,

$$\int_{N+1}^\infty f(x)dx < S - S_N < a_{N+1} + \int_{N+1}^\infty f(x)dx. \tag{15}$$

4.6 Leibnitz Criterion for alternating series

For an alternating series, $\sum(-1)^{n+1}a_n$ with $a_n > 1$, there is a simple test for conditional convergence. (This test does not make any claims on absolute convergence, which may be tested for with previously described tests.) If a_n is monotonically decreasing (for sufficiently large n) and $\lim_{n\to\infty} a_n = 0$, then the sum convergences. Additionally, the error in truncating the series is less than the value of the first term dropped, $S - S_N < a_{n+1}$.

5 Operations with series

1. $\sum ku_n = k \sum u_n$.
2. $\sum(u_n + v_n) = \sum u_n + \sum v_v$.
3. Addition or removal of a finite number of terms from a series does not affect ifs convergence.
4. If $\sum u_n$ and $\sum v_n$ are both absolutely convergent, then the Cauchy product $\sum w_n = \sum u_n \sum v_n$, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1, \tag{16}$$

is also absolutely convergent.