# Lecture 10: Forms and Traditional Vector Calculus

## 1 Differential Forms

Forms are antisymmetric, covarient tensors. A $p$-form has $p$ subscripted indicies, $\omega_{i_1 i_2 \ldots i_p}$ A zero form is defined to be a scalar, $\omega = f$. A one form is the usual differential form we talked about before, such as the differential of a scalar function, $\omega = df$.

To generate other forms, we need a wedge product with the following properties,

1. $\omega \wedge \nu = -\nu \wedge \omega$.
2. $(a\omega + b\mu) \wedge \nu = a\omega \wedge \nu + b\mu \wedge \nu$.
3. $\omega \wedge \omega = 0$.

Then two-forms can be any linear combination of wedge products of two one-forms. Similarly, $p$-forms are linear combinations of wedge products of $p$ one-forms. Note that an in an $n$-dimensional all $p$-forms with $p > n$ are necessarily zero. Also, all $p$-forms with $p = n$ are linear combinations of each other (this $p$-form is the $n$-dimensional volume element.

## 2 Turning forms into vectors in three dimensions

In three dimensional Euclidean space we have several structures that are useful for converting forms into vectors. The metric converts one forms into vectors, and allows us to define an oriented, normalized basis 3-vector. The oriented, normalized basis is a skew-symmetric product of normalized basis vectors,

$$\sigma^{ijk} = \sum_{lmn} \epsilon_{lmn} e_l^i e_m^j e_n^k. \qquad (1)$$

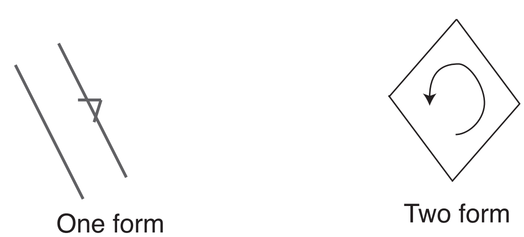


**Figure 1:** Sketch of a one-form and a two-form in two dimensions.

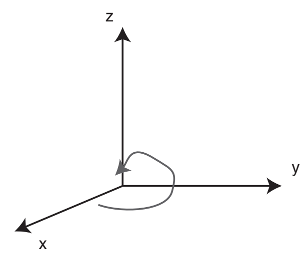


**Figure 2:** An oriented basis, $\sigma^{ijk}$.

Note that for $(e_1, e_2, e_3) = (\hat{x}, \hat{y}, \hat{z})$ the oriented, normalized basis has coordinates $\sigma^{ijk} = \epsilon^{ijk}$

In three dimensional euclidean space, you can contract with normalized, oriented vector basis set or the euclidean metric to form vectors.

## 2.1 0-forms to scalars

An example of a 0-form is a potential energy field. This is trivial, since 0-forms are already scalars.

## 2.2 1-forms to vectors

An example of a 1-form is a force. Contract with the metric to form a vector,

$$v^i = \sum_{ij} g^{ij} v_j. \tag{2}$$

## 2.3 2-forms to vectors

An example of a two-form is a current density. Contract the with a normalized vector basis set to form a vector.

$$J^i = \sum jk\sigma^{ijk} J_{ij}. \tag{3}$$

## 2.4 3-forms to scalars

An example of a 3-form is a charge density. Contract with a normalized vector basis set to form a scalar representing density.

# 3 Products of vectors

## 3.1 Dot product

$$(u \cdot v) = \sum_{ij} u_i g^{ij} u_j = \sum_i u^i v_j = \sum_i u_i v^j. \tag{4}$$

## 3.2 Cross product

$$(u \times v)^k = \sum_{ij} \sigma^{ijk} (u \wedge v)_{ij}. \tag{5}$$

## 3.3 Triple product

$$(u \cdot v \times w) = \sum_{ijk} \sigma^{ijk} u_i v_j w_k \tag{6}$$

# 4 Differential operators

The usual vector differential operators can be written in terms of the exterior derivative. The results are forms, which can than be transformed into scalars or vectors.

## 4.1 Gradient

$$(\nabla \cdot f)_i = df \tag{7}$$

## 4.2 Divergence

$$(\nabla \cdot u)_{ijk} = du_{ij} \tag{8}$$

## 4.3 Curl

$$(\nabla \times u)_{ij} = du_i \tag{9}$$