Lecture 13: Series of Functions

1 Uniform convergence

In this lecture we consider series of functions,

$$S(x) = \sum_{n=1}^{\infty} u_n(x); \qquad a \le x \le b. \tag{1}$$

The concepts of absolute and conditional convergence can be applied for any x in the range $[a, b]$. However, a series of function also allows for the notions of continuity, integration, and differntiation, which span several or all values of x in the range. It is therefore useful to introduce a new form of convergence to define functions that converge nicely when considered as functions.

A series of functions $S(x) = \sum u(x)$ is *uniformly convergent* in a range $[a, b]$ if it approaches a function everwhere in that range, not point by point, but globally. Mathmatically, we say that for any small $\epsilon > 0$ there is a $N(\epsilon)$, *independent of* x, such that

$$|S(x) - s_n(x)| < \epsilon. \tag{2}$$

Note that uniform convergence of a series of functions is a property independent of absolute convergence. Although the two properties may occur together, it is also possible to conditional uniform convergence, or non-unform absolute convergence.

1.1 Weierstrass M Test

If we construct a series of numbers $\sum_{n=1}^{\infty} M_n$ such that $M_n \ge |u_n(x)|$ for all x in $[a, b]$ and $\sum_{n=1}^{\infty} M_n$ is convergent, then the series will be uniformly convergent in $[a, b]$. Note that this also implies that the function series is absolutely convergent. (Since function series can be uniformly and conditionally convergent, this test does not work for all cases.)

1.2 Properties of uniformly convergent series

A series of functions $f(x) = \sum_{n=1}^{\infty} u_n(x)$ that is uniformly convergent in some interval $a < x < b$ has some useful properties,

1. If the individual terms are continuous, then the series sum is continuous.

2. If the individual terms are continuous, then the series may be integrated term by term. The sum of the integrals is the integral of the sum,
$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx. \tag{3}$$
3. The series may be differentiated term by term, provided the following additional criteria are met: $u_n(x)$ and $u_n'(x)$ are continuous in $a < x < b$, and $\sum_{n=1}^{\infty} u_n'(x)dx$ is uniformly convergent in (a, b).

2 Power series

A common expansion in physics is a power series,
$$P(x) = a_0 + a_1 x + +a_2 x^2 + \cdots . \tag{4}$$
Power series are useful for several reasons: (1) if x is small, then the series convergece quickly, (2) the power series is uniformly convergent in in interval of convergence, (3) differentiating of integrating a power series term-by-term does not change the interval of convergence, (4) power series representations of functions are unique, (5) power series form the basis for the theory of analytic functions (complex analysis).

Note that point 2, term-by-term differentiation and integration of a series is not a general property of series, but a particularly useful property of power series.

The D'Alebert ratio test is a particularly convenient way to find the interval of convergence of a power series. The convergence at the endpoints must be checked explicitly. Integration or differentiation may change the convergence properties of the endpoints.

3 Taylor's Series

A common way to find the power series for a function is to use the Taylor's series,
$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_{n+1}. \tag{5}$$
The remainder takes the form of a multidimensional integral,
$$R_n = \int_a^x \cdots \int f^{(n)}(dx)^n = \frac{(x-a)^n}{n!}f^{(n)}(\xi). \tag{6}$$
for some ξ between a and x.

If the Taylor series is expanded about $a = 0$, then it is known as Maclaurin's series.

4 Important series

4.1 Exponential function

$$\exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{for } -\infty < x < \infty. \tag{7}$$

4.2 Sine and consine functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{for } -\infty < x < \infty. \tag{8}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{for } -\infty < x < \infty. \tag{9}$$

These expansions show illustrate the relationships between e^{ix}, $\sin x$, and $\cos x$,

$$e^{ix} = \cos x + i \sin x \tag{10}$$

$$\cos x = \Re e^{ix} = \frac{e^{ix} + e^{-ix}}{2}, \tag{11}$$

$$\sin x = \Im e^{ix} = \frac{e^{ix} - e^{-ix}}{2i}. \tag{12}$$

The hyperbolic functions are given by $\cosh ix = \cos x$ and $\sinh ix = \sin x$,

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad \text{for } -\infty < x < \infty. \tag{13}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots . \tag{14}$$

These hyperbolic functions sinh and cosh are thus the symmetrized and antisymmetrized exponential function,

$$e^x = \cos x + \sin x \tag{15}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \tag{16}$$

$$\sinh x = \frac{e^x - e^{ix}}{2}. \tag{17}$$

4.3 Binomial Theorem

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots \qquad \text{for } -\infty < x < \infty. \tag{18}$$

Note that this series is a finite, nth order polynomial if n is a positive integer.

4.4 A couple other useful series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{for } -1 < x < 1. \tag{19}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \leq 1. \tag{20}$$