Lecture 17: Residue Theorem and Laurent Series

1 Cauchy integral theorem

For a function that is analytic on a simply connected region,

$$\oint_C f(z)dz = 0. \tag{1}$$

If the partial derivatives $\partial_x u$, $\partial_y u$, $\partial_x v$, and $\partial_y v$ are continuous, then Cauchy's integral theorem follows from Stoke's theorem in the plane and the Cauchy-Riemann conditions,

$$\begin{aligned}\oint_C f(z)dz &= \oint_C [u(x,y)dx - v(x,y)dy] + i\oint_C [v(x,y)dx + u(x,y)dy] \\ &= \iint [-\partial_y u(x,y) - \partial_x v(x,y)]dxdy + i\iint [-\partial_y v(x,y) + \partial_x u(x,y)]dxdy \\ &= \iint 0dxdy + i\iint_C 0dxdy \\ &= 0\end{aligned} \tag{2}$$

A consequence of this is that complex integrals in simply connected regions are independent of the choice of contour. When the analytic regions are not simply connected, the contour may be continuously deformed without changing the value of the integral.

Although this proof by Stoke's therem requires continuous partial derivatives, the theorem can be proved without that constraint.

If a closed contour circles several non-analytic regions (i.e. the analytic region is multiply connected, then the countour can be deformed to give

$$\oint_C f(z)dz = \sum_i \oint_{C_i} f(z)dz, \tag{3}$$

where the countours C_i each circle one of the non-analytic region.

2 Cauchy integral formula

Cauchy's integral formula determines the value of an analytic function in the interior of a simply connected region from its values on the contour,

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}dz. \tag{4}$$

From Cauchy's integral theorm, we see that we only need to prove this for an infintesimally small contour around z_0,

$$\begin{aligned}\oint \frac{f(z)}{z-z_0}dz &= \lim_{r\to 0}\int_0^{2\pi} \frac{z_0+re^{i\theta}}{re^{i\theta}} ire^{i\theta}d\theta \\ &= if(z_0)\int_0^{2\pi} d\theta \\ &= 2\pi i f(z_0).\end{aligned} \tag{5}$$

Cauchy's integral formula can also be used to determine the derivative of a function,

$$\begin{aligned}f'(z_0) &= \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\ &= \lim_{\Delta z\to 0}\frac{1}{2\pi i\Delta z}\left(\oint \frac{f(z)}{z-z_0-\Delta z}dz - \oint \frac{f(z)}{z-z_0}dz\right) \\ &= \lim_{\Delta z\to 0}\frac{1}{2\pi i\Delta z}\oint \frac{[(z-z_0)-(z-z_0-\Delta z)]f(z)}{(z-z_0-\Delta z)(z-z_0)}dz \\ &= \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}dz.\end{aligned} \tag{6}$$

In fact, derivatives of any order can be determined this way,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}dz. \tag{7}$$

This is an important distinction between differentiation of real functions and differentiation of complex functions. If a complex function is differentiable, then all of its higher order derivatives exist. Or, to put it another way, the derivative of an analytic function is another analytic function.

3 Review: Cauchy's Throrem and Cauchy's formula

Cauchy's theorm states that integrals around closed contours where the integrand is analytic in the interior have a value zero,

$$\oint_C f(z)dz = 0. \tag{8}$$

Cauchy's formula says that the value of an analytic function at some point z_0 is determined by it's values on a countour C encircling z_0,

$$f(z_0) = \frac{1}{2i\pi}\int_C \frac{f(z)}{z-z_0}dz. \tag{9}$$

Cauchy's formula can also be generalized to show that all derivatives (of any order) of an analytic function at point z_0 are determined by the function's values on a contour C encircling the point,

$$f^{(n)}(z_0) = \frac{n!}{2i\pi}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz. \tag{10}$$

4 Taylor's theorem for complex variables

We can use Cauchy's formula to prove Taylor's theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{11}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{12}$$

To see this, divide both sides of Eq. (11) by $(z - z_0)^{n+1}$, integrate around z_0, and apply Cauchy's theorem to both sides.

5 Laurent series

If a function f has a pole of order p at a point z_0, then the function $g(z) = (z - z_0)^p f(z)$ is analytic at z_0 and in a neighborhood of z_0. This prompts a generalization of Taylor's series to include posible divergence at z_0. The Laurent series is defined by

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n, \tag{13}$$

where

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz. \tag{14}$$

Note that Eq. (22) has been written to be valid for all n, although it reduces to Eq. (19) for analytic functions and non-negative n.

The terms in the Laurent series with $n \geq 0$ are called the analytic part, and the terms with $n < 0$ are the principle part. If the Laurent series has a finite number of terms in the principle part, it is said to have a pole of order p at z_0, where a_{-p} is the lowest non-zero coeficient. The residue of a pole is the value of the a_{-1} coeficient, (not the a_{-p} coefficient), and is used in the evaluation of integrals via the residue theorem.

6 The Residue Theorem

Complex integration is a powerful tool because of the residue theorem,

$$\oint_C f(z) = 2\pi i \times (\text{sum of residues inside of } C). \tag{15}$$

When a function is analytic inside of a countour, except at some poles, then the integral of the function around the countour is $2\pi i$times the sum of the residues the poles. The residues are the a_{-1} terms in the Laurent expansion at the pole — the weight of the $(z-z_0)^{-1}$ divergence at the pole.

7 How did we get here?

Cauchy's theorm states that integrals around closed contours where the integrand is analytic in the interior have a value zero,

$$\oint_C f(z)dz = 0. \tag{16}$$

This allows us to deform the contour on the left side of Eq. (15) so that it becomes a sum of contours around the poles. Cauchy's formula shows that all derivatives (of any order) of an analytic function at point z_0 are determined by the function's values on a contour C encircling the point,

$$f^{(n)}(z_0) = \frac{n!}{2i\pi}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz. \tag{17}$$

We can use Cauchy's formula to prove Taylor's theorem,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \tag{18}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{19}$$

The Laurent series generalizes this result to functions with a pole at $z+0$, and is defined by

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n, \tag{20}$$

where

$$a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}dz. \tag{21}$$

From Eq. (22) with $n=-1$ we see that

$$a_{-1} = \frac{1}{2\pi i}\oint f(z)dz, \tag{22}$$

which leads directly to the residue theorem, Eq. (15), where a_{-1} is called the residue.

8 Example: integrand decays as fast as z^{-2}

A simple example is the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}. \tag{23}$$

Since the integrand decays as fast as z^{-2}, we can add a integral that arcs across the upper half plane, conecting the $x = +\infty$ point to the $x = -\infty$. Then the integral becomes an integral around a closed contour,

$$I = \oint_C \frac{dz}{z^2+1}. \tag{24}$$

The integrand factors,

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}, \tag{25}$$

thus it has poles at $z = i$ and $z = -i$. The residues at these poles are

$$(z-i)f(z)|_{z=i} = (z-i)\frac{1}{(z-i)(z+i)}\bigg|_{z=i} = \frac{1}{2i}, \tag{26}$$

at $z = i$ and

$$(z-i)f(z)|_{z=-i} = (z-i)\frac{1}{(z-i)(z+i)}\bigg|_{z=-i} = \frac{-1}{2i}, \tag{27}$$

at $z = -i$.

Taking the contour around the upper half plane places the $z = i$ pole in the contour, so that

$$I = 2\pi i \times \frac{1}{2i} = \pi. \tag{28}$$

We can also take the contour in the lower half plane, in which case we have a *clockwise* contour and must add a negative sign. Then the $z = -i$ pole is containted in the contour, so

$$I = -2\pi i \times \frac{-1}{2i} = \pi. \tag{29}$$

Note that the choice of contour should not (and does not) affect the value of the integral for this problem. (In the next lecture we discuss cases where the poles lie *on* the contour, and the choice of contour will lead to different values of the integral. The choice of contour can only affect the value of the integral if the original integral on the real axis is itself not uniquely defined.)

9 Example: integrand has $e^{i\alpha x}$ with $(\alpha > 0)$

The function $e^{i\alpha z}$ goes to zero as $\Im z$ gets larger, so the countour should be closed in the upper half plane. (Conversely, integrals with $e^{-i\alpha z}$ should be closed in the lower half plane.) Provided that the integrand vanishes as $|z| \to \infty$, the closing contour doesn't change the value of the integral. Then the integrand becomes a contour integral,

$$I = \oint_C f(z) e^{i\alpha z} dz. \tag{30}$$

For example, if $f(z) = \frac{1}{z^2+1}$, we have poles at $z = i$ and $z = -i$ as in the previous example. The residue at $z = i$ is

$$(z-i)f(z)|_{z=i} = (z-i)\frac{e^{i\alpha z}}{(z-i)(z+i)}\bigg|_{z=i} = \frac{e^{-\alpha}}{2i}, \tag{31}$$

so $I = \pi e^{-\alpha}$.