# Lecture 31: Separation of Variables

## 1 Separable equations

Consider a linear partial differential equations (PDE's)

$$Lu(x, y, z, t) = 0, \tag{1}$$

or

$$Lu(x, y, z) = \lambda u(x, y, z), \tag{2}$$

where $L$ is a linear differential operator. If $L$ can be written as a sum of linear operators,

$$L = L_x + L_y + L_z + L_t, \tag{3}$$

where $L_x$ contains only the variable $x$ and the derivative operators of $x$ (generally $\partial x$ and $\partial^2 x$), and likewise for $L_y$, $L_z$, and $L_t$, then the PDE is seperable. (We can relax this condition a little more, see examples with cylindrical and spherical coordinates below.) By writing $u(x, y, z, t) = X(x)Y(y)Z(z)T(t)$ and dividing the PDE by $u$, we get

$$\frac{L_x X(x)}{X(x)} + \frac{L_y Y(y)}{Y(y)} + \frac{L_z Z(z)}{Z(z)} + \frac{L_t T(t)}{T(t)} = 0 \qquad (\text{or} = \lambda). \tag{4}$$

For constants $n_x + n_y + n_z + n_t = 0$ (or $n_x + n_y + n_z + n_t = \lambda$), the PDE splits into ODE eigenvalue equations,

$$L_x X(x) = n_x X(x) \tag{5}$$

$$L_y Y(y) = n_y Y(y) \tag{6}$$

$$L_z Z(z) = n_z Z(z) \tag{7}$$

$$L_t T(t) = n_t T(t). \tag{8}$$

## 2 Laplacian in Rectangular, Cylindrical, and Spherical Coordinates

Many physical equations get their spatial dependence from the Laplacian. The Laplacian operator is separable in several coordinate systems:

- Rectanglular coordinates,

$$\nabla^2 u(x, y, z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{9}$$

- Cylindrical coordinates,

$$\nabla^2 u(\rho,\theta,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}. \tag{10}$$

- Spherical coordinates,

$$\nabla^2 u(x,y,z) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \tag{11}$$

Therefore, many common PDEs can be separated into three ODEs in several common coordinate systems. These ODEs can be put in Sturm-Liouville form, and this gives rise to several sets of special functions, such as sine and cosine functions, Legendre polynomials, associated Legendre polynomials, and Bessel functions. Also, other terms in the PDE, such as the potential term in Schrödinger's equation, may be compatible with separation of varialbes, which leads to other equations. For example, the potential in the hydrogen atom has the form $1/r$, which is compatible with separation of variables in spherical coordinates. This extra term changes the radial equation of the Schrödinder equation from a Bessel equation to a Laguerre equation, introducing Laguerre polynomials into the wavefunctions for hydrogen atom orbitals.

## 3 Example: Heat dissipation on a square plate

Heat flow in the presence of a temperature gradient is given by the thermal conductivity, $\kappa$,

$$\mathbf{Q} = -\kappa\nabla T. \tag{12}$$

Since heat and temperature are related by $dq = C_v dT$, conservation of energy gives the heat equation,

$$C_v\frac{\partial T}{\partial t} = \kappa\nabla^2 T, \tag{13}$$

which is just the diffusion equation with diffisuion constant $D = \kappa/C_V$. Even though $C_V$ and $\kappa$ could both depend on temperature, for this example we will assume that they are temeperature independent (otherwise the heat equation becomes non-linear).

For this example, we will solve the heat equation, Eq. (13), on a unit sqaure with boundaries held at $T = 0$. The initial condition will be $T(x,y,t=0) = T_0\delta(x-x_0)\delta(y-y_0)$. Physically, this would be a plate or tile at thermal equilibrium ($T = 0$ corresponds to some reference temperature), which has had the temperature at one point suddenly increased to $T_0$ at a point due to radiation, or an inelastic impact, or perhaps the attachment of a small, hot mass (small enough to neglect the heat capacity of the additional mass). The expected solution is for the heat to spread from the point $(x_0,y_0)$ (as a broadening Gaussian), and flow off the edges of the plate. After a long time, the plate will return to thermal equilibrium, $\lim_{t\to\infty} T(x,y,t) = 0$.

To solve this, we first use separation of variables to reduce the probem to three ODE's. Since the boundary conditions are rectangular, we will use cartesian coordinates,

$$T(x,y,t) = X(x)Y(x)S(t). \tag{14}$$

The boundary conditions that $T = 0$ on the boundary can be implemented by requiring $X(0) = X(1) = 0$ and $Y(0) = Y(1) = 0$.

The heat equation in rectangular coordinates is

$$C_v \partial_t T = \kappa \delta_x^2 T + \kappa \delta_y^2 T. \tag{15}$$

This separates into three ODE with separation constants $n_x$ and $n_y$,

$$\frac{C_v}{\kappa} S' = -\pi^2 (n_x^2 + n_y^2) S \tag{16}$$

$$X'' = -\pi^2 n_x^2 X \tag{17}$$

$$Y'' = -\pi^2 n_y^2 Y \tag{18}$$

The separation constants have been chosen work well with the boundary conditions. (In practice, use separation constants $\lambda_x$ and $\lambda_y$, solve the $X$ and $Y$ equations using the boundary conditions, then define $n_x$ and $n_y$ to have convenient integer values.) These solutions, subject to the boundary conditions, are

$$T_{n_x,n_y}(x,y,t) = \sin(n_x \pi x) \sin(n_y \pi y) \exp\left[ -\frac{\kappa \pi^2 (n_x^2 + n_y^2)}{C_v} t \right]. \tag{19}$$

The general solution is

$$T(x,y,t) = \sum_{n_x,n_y} a_{n_x,n_y} T_{n_x,n_y}(x,y,t), \tag{20}$$

with $a_{n_x,n_y}$ determined by the temperature at $t = 0$. For $T(x,y,t) = \delta(x - x_0)\delta(y - y_0)$ we find $a_{n_x,n_y} = 4 \sin n_x \pi x_0 \sin n_y \pi y_0$.