Lecture 7: Systems of Linear Equations

1 Systems of linear equations

Consider a system of N linear equations with N unknowns,

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 \cdots + A_{1N}x_N &= b_1 \\ A_{21}x_1 + A_{22}x_2 \cdots + A_{2N}x_N &= b_2 \\ &\vdots \\ A_{N1}x_1 + A_{N2}x_2 \cdots + A_{NN}x_N &= b_N. \end{aligned} \tag{1}$$

This is equivalent to a vector equation with a linear operator,

$$A\mathbf{x} = \mathbf{b}. \tag{2}$$

1.1 Solution using Matrix Inverse

Multiplying the set of equations by the inverse of A, we have

$$\mathbf{b} = A^{-1}\mathbf{x}. \tag{3}$$

Recall that the matrix inverse may be calculated by

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det |A|}, \tag{4}$$

which is a result derived in the text.

1.2 Solution by application of operations

Another way to solve this problem by hand is to repeatedly apply the same linear operators to both sides of Eq. (2). A conviention choice of operations are:

- Switch two rows, i.e.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

- Scale a row, i.e.
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

- Add a row to another row, i.e.
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

This is also a way to find an inverse without computing determinants.

1.3 Solution by Crammer's rule

To derive Crammer's rule, start from the determinant of A,

$$\det|A| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}. \tag{8}$$

Add columns two and three to column one, scaled by (x_2/x_1) and (x_3/x_1), respectively (determinant is unchanged). Then multiply column one by x_1 (determinant is scaled by x_1). Then we have

$$\det|A| = \frac{1}{x_1} \begin{vmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 & A_{12} & A_{13} \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 & A_{22} & A_{23} \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 & A_{32} & A_{33} \end{vmatrix} = \frac{1}{x_1}\Delta_1. \tag{9}$$

But the first colum of this matrix is just (b_1, b_2, b_3), so the solution to the equations is

$$x_1 = \frac{\Delta_1}{|A|}, \qquad x_2 = \frac{\Delta_2}{|A|}, \qquad x_3 = \frac{\Delta_3}{|A|}, \tag{10}$$

where the Crammer's matrices are

$$\Delta_1 = \begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} A_{11} & b_1 & A_{13} \\ A_{21} & b_2 & A_{23} \\ A_{31} & b_3 & A_{33} \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ A_{31} & A_{32} & b_3 \end{vmatrix}. \tag{11}$$

2 Worked Example

Solve the set of equations,

$$\begin{aligned} 2x_1 + 4x_2 &+ 3x_3 = 4 \\ x_1 - 2x_2 &- 2x_3 = 0 \\ -3x_1 + 3x_2 &+ 3x_3 = -7, \end{aligned} \tag{12}$$

using each of the three techniques. We find $x_1 = 2$, $x_2 = -3$, and $x_3 = 4$, using $\det|a| = 11$, $\Delta_1 = 22$, $\Delta_2 = -33$, $\Delta_3 = 44$, and

$$A^{-1} = \frac{1}{11} \begin{vmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & 8 \end{vmatrix}. \tag{13}$$

3 Special square matricies

Diagonal, symmetric, antisymmetric, orthogonal, and Hermetian matricies will be discussed later.

4 Eigenvectors and Eigenvalues of Normal Matricies (we will cover this later this semester)

Normal matricies commute with their Hermetian conjugate, $AA^\dagger = A^\dagger A$. Normal matricies include Hermetian matricies ($A^\dagger = A$) and unitary matricies ($A^\dagger = A^{-1}$), as well as the corresponding real valued matricies (symmetric and orthogonal matricies).

Eigenvectors are special vectors (directions) that get scaled by the matrix. The scale factor each eigenvector is the eigen value. Mathematically,

$$Av = \lambda v, \tag{14}$$

where A is a matrix and λ is a scalar eigenvalue. There are a few important results for the eigenvectors of a normal matrix:

- All eigenvalues of a Hermetian (or symmetric) matrix are real.
- Eigenvectors with distinct eigenvalues are orthogonal.
- All eigenvectors of normal matricies can be made orthogonal.
- All eigenvalues of a unitary (or orthogonal) matrix have modulus one.

5 Determining eigenvalues and eigenvectors: the characteristic equation

The eigenvalues are zeros of the *characteristic determinant*. To see this construct a new matrix $A - \lambda I$ that is a function of the parameter λ. Then

$$\begin{aligned} (A-\lambda)v &= 0 \\ \det|A-\lambda| &= 0. \end{aligned} \tag{15}$$

Once the eigenvalues are found, the eigenvectors can be solved for.

6 Similarity transformations

If § is a unitary matrix, the matrix A may be transformed as

$$A' = S^{-1}AS. \tag{16}$$

This is the matrix equivalent of a change of basis. Several properties are unchanged:

- The value of the determinant is unchanged.
- The value of the trace is unchanged.
- The identity matrix is unchanged by a unitary transform .
- The characteristic is unchanged, so that the eigenvalues are unchanged.

A useful similarity transform is the matrix of eigenvectors. This transform is said to *diagonalize* the matrix, for example,

$$A' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{17}$$

From this we get two very important results:

- The determinant is the product of the eigenvalues.
- The trace is the sum of the eigenvalues.

7 Examples

7.1 Eigenvalues and eigenvectors of a two-dimensional rotation

The two-dimensional rotation matrix is

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{18}$$

We could write the characteristic determinant, set it to zero, and solve. However, in this case we know that $\det|R| = \cos^2\theta + \sin^2\theta = 1$ and $TrR = 2\cos\theta$. Thus $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = 2$, so that $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$. Note that these eigenvalues are only real for the special cases $\theta = 0$ and $\theta = \pi$. The eigenvectors can be readily solved for. In particular, when $\theta = \pi$, the eigenvectors are $(1,1)$ and $(1,-1)$ for eigenvalues $\lambda = -1$ and $\lambda = 1$, respectively.

7.2 Normal modes

For three objects with masses m, $2m$, and m, arranged linarly with springs having spring constant k, the energy is

$$L = \frac{1}{2}\dot{x}T\dot{x} - \frac{1}{2}xVx, \tag{19}$$

where

$$T = \begin{pmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{pmatrix}, \tag{20}$$

and

$$V = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}, \tag{21}$$

and the object displacements are given by the vector x. Solutions of the form $x(t) = e^{iwt}x(0)$ exist if

$$\det|V - \omega^2 T| = 0. \tag{22}$$

The three solutions are $x_1 = (111)$, $x_2 = (101)$ and $x_3 = (1-11)$. These correspond to a uniform translation with $\omega = 0$, a symmetric vibration with $\omega = \sqrt{k/m}$, and an antisymmetric vibration with $\omega = \sqrt{2k/m}$. Notice that this is an eigenvalue problem in the coordinate system with $T' = 1$ (weighted coordinates).