Lecture 7: Introduction to Manifolds

1 Surfaces embedded in three dimensions

Before we start the description of manifolds, let's first look at the more familiar case of two-dimensional surfaces embedded in three dimensions. Such a surface may be defined as the set of zeros of a scalar function,

$$g(x, y, z) = 0. \tag{1}$$

Often it is useful to parameterize this surface with coordinates u and v, so that points on the surface are given by $\mathbf{r}(u, v)$ for a suitable range of u and v. For example, the surface of a sphere of radius a is defined as

$$\mathbf{r}(\theta, \phi) = \hat{x}a \sin\theta \cos\phi + \hat{y}a \sin\theta \sin\phi + \hat{z}a \cos\theta, \tag{2}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Note that this parameterization is not one-to-one at the poles; i.e., all points $(\theta, 0)$ map to $\mathbf{r} = a\hat{z}$.

The coordinates define two vectors at any point on the surface, $\partial_u\mathbf{r}$ and $\partial_v\mathbf{r}$. It is common to normalize these vectors, and for the case of a sphere we have

$$\hat{\theta} = \hat{x} \cos\theta \cos\phi + \hat{y} \cos\theta \sin\phi - \hat{z}sin\theta, \tag{3}$$

$$\hat{\phi} = -\hat{x} \sin\phi + \hat{y} \cos\phi. \tag{4}$$

These vectors form the basis for the tangent plane at that point. A third vector, normal to the plane, may be defined as $\hat{n} = \hat{u} \times \hat{v}$. For the sphere, this normal vector is

$$\hat{r} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta. \tag{5}$$

2 Manifolds

Additional material may be found in other texts, such as Spivak (1965), Burke (1985), Schutz (1980), and Ch. 5 of Nakahara (1990).

The definition of a surface as a subset of R^3 is sometimes to restrictive. Manifolds generalize the concept of a surface. An *N-dimensional manifold* is a space that locally appears like R^N, which is smooth enough allow calculus. The surfaces described in the previous section are all 2-d manifolds. Other 2-d manifolds cannot be embeded in three dimensions; for example, a mobius strip or a Klein bottle.

2.1 Coordinate patches

An n-dimensonal manifold may be covered by several coordinate systems $(x_1, x_2, \ldots, x_n)$, $(x'_1, x'_2, \ldots, x'_n)$, etc. In general, one coordinate system will not suffice. For example, polar coordinates describe a sphere at all points except the poles. Other coordinate systems, such as stereographic coordinates ($X = \frac{x}{R-z}, Y = \frac{y}{R-z}$) are needed to accurately describe the poles. A mobius strip needs multiple coordinate systems, which are then patched together so that it gets one twist as you go around it.

3 Tangent Spaces and Vector Fields

The unit vectors $(\hat{x_1}, \hat{x_2}, \ldots \hat{x_n})$ form a basis for an n-dimensional vector space at any point on the manifold, T_p. Another point p' has its own tangent space, $T_{p'}$. Selecting one vector from the tangent space of each point in the manifold give a vector field over the manifold.

3.1 Why separate vector spaces at each point?

Sometimes you can get away with thinking of vector spaces as function from the manifold to one vector space. However, it is often impossible to identify the tangent space at one point with the tangent space at another point. For example, on the surface of the earth, we might try to identify north at one point with north at another point. Although this distinction is reasonable for points close together (Pheonix and Tuscon), it becomes tricky for points far apart or near the poles. (Imagine trying to build a house at the north pole with the assumption that north, east, west, and south directions were useful concepts of direction at the pole!). This is not just a coordinate problem: If you head north from the equator, turn right at the north pole, and turn right at the equator, you can return to the same point and show that *two* rights can make a left — on a sphere.

3.2 Not all vector fields are in tangent spaces

In E&M we often describe electric or magnetic fields at surfaces. These fields are *not* in the surfaces tangent space: they are three dimensional vectors, and the surface is two dimensional. Vectors at different points can be compared several diffent ways, either as three dimensional vectors, or by comparing their normal components.

Mathematically, these three dimensional vectors are often decomposed into a normal component (a scalar field on the 2-d manifold) and the tangental component (which is a vector field in the tangent space).

4 Tangent spaces as local view of a curve

The tangent vector space on a manifold follows naturally from curves. A curve $\mathbf{r}(s)$ is a map $R \rightarrow M$ from the real line points on the manifold. The set of all possible velocities of curves through a point is the tangent space at that point.

5 Differential one-forms and line integrals

Recall that differential forms were described as the local (linear) behavior of a function. This definition works well for manifolds. Equivalently, these differential forms at a point are the covectors to the tangent space. To see this, consider a tangent vector $\mathbf{v}$ and a form ω. This tangent vector lies on a curve $\mathbf{r}(s)$, and the form is part of a function $f(\mathbf{r})$ (there are many possible curves and functions, just choose one). The contraction of the tangent vector and the form is a number: df/ds, the rate of change of the function f along the curve.

We can formally denote line integrals over forms as

$$\int_C \omega. \tag{6}$$

In practice, you use coordinates, and this integral becomes a ordinary one-dimensional integral. This more formal notation of the integral of a form is useful when we will talk about muliple-dimensional integrals and higher-order forms (corresponding to surface and volume integrals in three dimensional vector calculus).

References

Burke, W., *Applied Differential Geometry* (Cambridge Univ. Press, Cambridge, 1985).

Nakahara, M., *Geometry, Topology and Physics* (Institute of Physics Publishing, London, 1990).

Schutz, B., *Geometric Methods of Mathematical Physics* (Cambridge Univ. Press, Cambridge, 1980).

Spivak, M., *Calculus on Manifolds* (Addison-Wesley, Reading Mass., 1965).